Math 217 Fall 2025
Quiz 19 – Solutions

Dr. Samir Donmazov

1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Let $\mathfrak{B} = (v_1, \ldots, v_d)$ be a basis for the vector space V. Let $v \in V$. The $\mathfrak{B}$-*coordinates* of v are ...

   Solution: The unique scalars $a_1, \ldots, a_d \in \mathbb{R}$ (or in the base field) such that

   $$v = a_1 v_1 + \cdots + a_d v_d.$$

   (b) Let $\mathfrak{B} = (v_1, \ldots, v_d)$ be a basis for the vector space V. Let $v \in V$. The $\mathfrak{B}$-*coordinate column vector* of v is ...

   Solution: The column vector formed by the $\mathfrak{B}$-coordinates of v:

   $$[v]_{\mathfrak{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} \quad \text{where } v = a_1 v_1 + \cdots + a_d v_d.$$

   (c) Suppose X and Y are sets. A function $f : X \to Y$ is called *surjective* provided that ...

   Solution: For every $y \in Y$ there exists $x \in X$ such that $f(x) = y$; equivalently, $\mathrm{im}(f) = Y$.

2. Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   (a) Let $\vec{w}$ be the vector whose $\mathfrak{B}$-coordinates are $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. What is $[\vec{w}]_{\mathcal{E}}$?

   Solution:

   $$\vec{w} = 2\vec{v}_1 + (-1)\vec{v}_2 = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

   Hence $[\vec{w}]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

(b) Let $\vec{v}$ be such that $[\vec{v}]_{\mathfrak{B}} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Represent $\vec{v}$ in standard coordinates. Find $[\vec{e}_1]_{\mathfrak{B}}$ and $[\vec{e}_2]_{\mathfrak{B}}$.

Solution:

$$\vec{v} = \tfrac{1}{2}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2 = \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{e}_2.$$

To find $[\vec{e}_1]_{\mathfrak{B}}$, solve $a\vec{v}_1 + b\vec{v}_2 = \vec{e}_1$:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow a = \tfrac{1}{2},\ b = -\tfrac{1}{2},$$

so $[\vec{e}_1]_{\mathfrak{B}} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$. Similarly, for $\vec{e}_2$:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow a = b = \tfrac{1}{2},$$

so $[\vec{e}_2]_{\mathfrak{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ (consistent with $\vec{v} = \vec{e}_2$ above).

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) If $V = \mathbb{R}^{2\times 2}$ with basis $\mathfrak{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$, then the $\mathfrak{B}$-coordinates of $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ are $\begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: FALSE. If $A = c_1B_1 + c_2B_2 + c_3B_3 + c_4B_4$ (in the listed order), matching entries gives

$$\begin{aligned} a_{11} &= c_1 + c_2 + c_3 + c_4 = 2, \\ a_{12} &= c_2 + c_3 + c_4 = 2, \\ a_{21} &= c_3 + c_4 = 3, \\ a_{22} &= c_4 = 4. \end{aligned}$$

Solving: $c_4 = 4$, $c_3 = -1$, $c_2 = -1$, $c_1 = 0$. Thus

$$[A]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 4 \end{bmatrix}.$$